Multifractal Analysis of inhomogeneous Bernoulli products

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Abstract We are interested to the multifractal analysis of inhomogeneous Bernoulli products which are also known as coin tossing measures. We give conditions ensuring the validity of the multifractal formalism for such measures. On another hand, we show that these measures can have a dense set of phase transitions.

Keywords : Hausdorff dimension, multifractal analysis, Gibbs measure, phase transition.

1 Introduction

Let us consider the dyadic tree $\mathbb{T}$ (even though all the results in this paper can be easily generalised to any ℓ-adic structure, $\ell \in \mathbb{N}$), let $\Sigma = \{0,1\}^{\mathbb{N}}$ be its limit (Cantor) set and denote by $(\mathcal{F}_n)_{n\in\mathbb{N}}$ the associated filtration with the usual $0-1$ encoding.

For $\epsilon_1, ..., \epsilon_n \in \{0,1\}$ we denote by $I_{\epsilon_1 \ldots \epsilon_n}$ the cylinder of the nth generation defined by $I_{\epsilon_1 \ldots \epsilon_n} = \{x = (i_1, ..., i_n, i_{n+1}, ...) \in \Sigma, \ ; \ i_1 = \epsilon_1, ..., i_n = \epsilon_n\}$. For every $x \in \Sigma$, $I_n(x)$ stands for the cylinder of $\mathcal{F}_n$ containing x.

If $(p_n)_n$ is a sequence of weights, $p_i \in (0,1)$, we are interested in Borel measures μ on σ defined in the following way

$$\mu(I_{\epsilon_1 \ldots \epsilon_n}) = \prod_{j=1}^{n} p_j^{1-\epsilon_j}(1-p_j)^{\epsilon_j}. \tag{1}$$

This type of measure will be referred to as an *inhomogeneous Bernoulli product*. The aim of this paper is to study multifractal properties of such measures.

The particular case where the sequence (p_n) is constant is well-known and provides an example of measure satisfying the multifractal formalism (see e.g [Fal97]). In the general case, Bisbas in [Bis95] gave a sufficient condition on the sequence (p_n) ensuring that μ is a multifractal measure (i.e. the level sets are not empty) . However, the work

of Bisbas does not provide the dimension of the level sets E_α associated to the measure μ.

Let us give a brief description of multifractal formalism. For a probability measure m on Σ, we define the *local dimension* (also called Hölder exponent) of m at $x \in \Sigma$ by

$$\alpha(x) = \liminf_{n\to+\infty} \alpha_n(x) = \liminf_{n\to+\infty} -\frac{\log m(I_n(x))}{n \log 2}.$$

The aim of multifractal analysis is to find the Hausdorff dimension, $\dim(E_\alpha)$, of the level set $E_\alpha = \{x : \alpha(x) = \alpha\}$ for $\alpha > 0$. The function $f(\alpha) = \dim(E_\alpha)$ is called the *singularity spectrum* (or multifractal spectrum) of m and we say that m is a *multifractal measure* when $f(\alpha) > 0$ for several $\alpha' s$.

The concepts underlying the multifractal decomposition of a measure go back to an early paper of Mandelbrot [Man74]. In the 80's multifractal measures were used by physicists to study various models arising from natural phenomena. In fully developped turbulence they were used by Frisch and Parisi [FP85] to investigate the intermittent behaviour in the regions of high vorticity. In dynamical system theory they were used by Benzi et al. [BPPV84] to measure how often a given region of the attractor is visited. In diffusion-limited aggregation (DLA) they were used by Meakin et al. [MCSW86] to describe the probability of a random walk landing to the neighborhood of a given site on the aggregate.

In order to determine the function $f(\alpha)$, Hentschel and Procaccia [HP83] used ideas based on Renyi entropies [Rén70] to introduce the generalized dimensions D_q defined by

$$D_q = \lim_{n\to+\infty} \frac{1}{q-1} \frac{\log\left(\sum_{I\in\mathcal{F}_n} m(I)^q\right)}{n \log 2},$$

(see also [GP83, Gra83]). From a physical and heuristical point of view, Halsey et al. [HJK+86] showed that the singularity spectrum $f(\alpha)$ and the generalized dimensions D_q can be derived from each other. The Legendre transform turned out to be a useful tool linking $f(\alpha)$ and D_q. More precisely, it was suggested that

$$f(\alpha) = \dim(E_\alpha) = \tau^*(\alpha) = \inf(\alpha q + \tau(q),\ q \in \mathbb{R}), \tag{2}$$

where

$$\tau(q) = \limsup_{n\to+\infty} \tau_n(q) \quad \text{with} \quad \tau_n(q) = \frac{1}{n \log 2} \log\left(\sum_{I\in\mathcal{F}_n} m(I)^q\right).$$

(The sum runs over the cylinders I such that $m(I) \neq 0$.) The function $\tau(q)$ is called the L^q-spectrum of m and if the limit exists $\tau(q) = (q-1)D_q$.

Relation (2) is called the multifractal formalism and in many aspects it is analogous to the well-known thermodynamic formalism developed by Bowen [Bow75] and Ruelle [Rue78].

For number of measures, relation (2) can be verified rigorously. In particular, if the sequence (p_n) is constant or periodic, the measure μ given by (1) satisfies the multifractal formalism (e.g. [Fal97]). Moreover some rigourous results have already been obtained for some invariant measures in dynamical systems (e.g [Col88, Fan94, Ran89]), for some self-similars measures under separation conditions (e.g [CM92, LN99, Ols95]) and for quasiindependent measures(e.g [BMP92, Heu98, Tes06]).

The minoration of $\dim(E_\alpha)$ usually follows on the existence of a shift-invariant and ergodic measure m_q (the so-called *Gibbs measure* [Mic83]), satisfying

$$\forall n,\ \forall I \in \mathcal{F}_n, \quad \frac{1}{C} m(I)^q 2^{-n\tau(q)} \leq m_q(I) \leq C m(I)^q 2^{-n\tau(q)},$$

where the constant $C > 0$ is independent of n and I. If τ is differentiable at q, the measure m_q is supported by $E_{-\tau'(q)}$ and Brown, Michon and Peyrière established [BMP92, Pey92] that

$$\dim(E_{-\tau'(q)}) = \tau^*(-\tau'(q)) = -q\tau'(q) + \tau(q).$$

If the weights p_n are not all the same, the measure μ is in general no shift-invariant and we cannot apply classical tools of ergodic theory, as Shannon-McMillan theorem (e.g [Bil65]), to get a lower bound of $\dim(E_\alpha)$.

Let us introduce the other following level sets defined by

$$\underline{E}_\alpha = \{x\ ;\ \alpha(x) \leq \alpha\}\,,\ \overline{F}_\alpha = \left\{x\ ;\ \limsup_{n\to\infty} \alpha_n(x) \geq \alpha\right\},$$

and

$$F_\alpha = \left\{x\ ;\ \limsup_{n\to\infty} \alpha_n(x) = \alpha\right\}.$$

We can now state our main results. In section 2, we prove the following.

Theorem 1.1 *Let μ be an inhomogeneous Bernoulli product on Σ and $q \in \mathbb{R}$. We have*

$$\liminf_{n\to\infty} -q\tau'_{\mu,n}(q) + \tau_{\mu,n}(q) \leq \dim\left(\underline{E}_{-\tau'(q^-)} \cap \overline{F}_{-\tau'(q^+)}\right) \leq \inf\left\{\tau^*(-\tau'(q^+)), \tau^*(-\tau'(q^-))\right\}.$$

The proof of the lower bound relies on the construction of a special inhomogeneous Bernoulli product which has the dimension of the level set studied.

In section 3 we are interested to the case where the sequence $\tau_{\mu,n}(q)$ converges. In this situation, we prove that the multifractal formalism holds for $\alpha = -\tau'_\mu(q)$–if it exists. More precisely, we have

Theorem 1.2 *Suppose that the sequence $(\tau_{\mu,n}(q))$ converges at a point $q > 0$. If $\tau'_\mu(q)$ exists and if $\alpha = -\tau'_\mu(q)$, we have*

$$\dim (E_\alpha \cap F_\alpha) = \tau^*_\mu(\alpha) = \alpha q + \tau_\mu(q). \tag{3}$$

Theorem 1.2 lead us to study the differentiability of the L^q-spectrum $\tau_\mu(q)$. A point q will be called a *phase transition* if $\tau'_\mu(q)$ does not exist. In section 4, we are interested to the existence of phase transitions. More precisely, we prove the following.

Theorem 1.3 *There exist inhomogeneous Bernoulli products μ presenting a dense set of phase transitions.*

2 Proof of theorem 1.1

We begin by a preliminary result.

Lemma 2.1 *If μ is an inhomogeneous Bernoulli product, then $(\tau''_{\mu,n})$ are locally uniformly bounded on $(0, +\infty)$.*

Proof We denote by $\beta(p_i)$ the Bernoulli homogeneous measure of parameter p_i and by $\tau(p_i, q)$ it's τ function, $\tau(p_i, q) = \log(p_i^q + (1-p_i)^q)$. Using the fact that μ is the product of $\beta(p_i)$ we easily obtain

$$\tau_{\mu,n}(q) = \frac{1}{n}\sum_{i=0}^{n} \tau(p_i, q) \quad q > 0.$$

It is therefore sufficient to show that ,for any $q_0 > 0$, there exists a constant $C = C(q_0)$ such that for all $p \in (0,1)$ and all $q > q_0$, $\dfrac{\partial^2 \tau(p,q)}{\partial q^2} \leq C$. The proof is straigthforward :

$$\begin{aligned}
\frac{\partial^2 \tau(p,q)}{\partial q^2} &= \frac{(p^q(\log p)^2 + (1-p)^q(\log(1-p))^2)}{(p^q + (1-p)^q)} - \frac{(p^q \log p + (1-p)^q \log(1-p))^2}{(p^q + (1-p)^q)^2} \\
&= \frac{p^q(1-p)^q\left((\log p)^2 + (\log(1-p))^2 - 2\log p \log(1-p)\right)}{(p^q + (1-p)^q)^2} \\
&= \frac{p^q(1-p)^q\left(\log \frac{p}{1-p}\right)^2}{(p^q + (1-p)^q)^2} \leq [4p(1-p)]^q (\log p)^2 \leq [4p(1-p)]^{q_0} (\log p)^2,
\end{aligned}$$

which is uniformly bounded on $p \in (0,1)$ and the proof is complete. •

Lemma 2.1 allows us to give estimates for the lower and the upper Hausdorff dimension of the measure μ. They are respectively defined by

$$\dim_*(\mu) = \inf\{\dim(E),\ \ \mu(E) > 0\}\ ;\ \dim^*(\mu) = \inf\{\dim(E),\ \ \mu(E) = 1\}.$$

We say that μ is exact if $\dim_*(\mu) = \dim^(\mu)$ and we note $\dim(\mu)$ the common value. In the same way, we can define the lower and the upper packing dimension of the measure μ. It is well known that there exist some relations between these quantities and the derivatives of the function $\tau_\mu(q)$ at $q = 1$. More precisely, it is proved in [Fan94, Heu98] that

$$-\tau'_\mu(1+) \leq \dim_*(\mu) \leq h_*(\mu) \leq h^*(\mu) \leq \mathrm{Dim}^*(\mu) \leq -\tau'_\mu(1-),$$

where $h_*(\mu)$ and $h^*(\mu)$ stand for the lower and the upper entropy of the measure μ, defined as

$$h_*(\mu) = \liminf -\frac{1}{n\log 2}\sum_{I\in\mathcal{F}_n}\mu(I)\log\mu(I) = \liminf -\tau'_{\mu_n}(1)$$

and

$$h^*(\mu) = \limsup -\frac{1}{n\log 2}\sum_{I\in\mathcal{F}_n}\mu(I)\log\mu(I) = \limsup -\tau'_{\mu_n}(1).$$

By Lemma 2.1, we deduce (see [BH02, Heu98]) the following remark.

Remark 2.2 If μ is an inhomogeneous Bernoulli product then

$$\dim\mu = \liminf_{n\to\infty} -\tau'_{\mu_n}(1) = -\tau'_\mu(1^+) = h_*(\mu).$$

and

$$\mathrm{Dim}\,\mu = \limsup_{n\to\infty} -\tau'_{\mu_n}(1) = -\tau'_\mu(1^-) = h^*(\mu).$$

Fix $q \in \mathbb{R}$. To prove theorem 1.1, we construct an auxiliary measure ν supported by the set $\underline{E}_{-\tau'(q^-)} \cap \overline{F}_{-\tau'(q^+)}$. More precisely, we consider a sequence of measures ν_n satisfying

$$\nu_n(I) = \frac{\mu(I)^q}{\sum_{I\in\mathcal{F}_n}\mu(I)^q} = \mu(I)^q|I|^{\tau_{\mu,n}(q)},$$

if $I \in \mathcal{F}_n$. The following lemma implies that the sequence (ν_n) converges in the weak* sense to a probability measure ν which is also an inhomogeneous Bernoulli product.

Lemma 2.3 *Let $n \in \mathbb{N}$ and $I \in \mathcal{F}_n$. If μ is an inhomogeneous Bernoulli product, we have $\nu_n(I) = \nu_{n+1}(I)$.*

Proof Take $n > 0$ and $I \in \mathcal{F}_n$. We can compute

$$\nu_{n+1}(I) = \frac{\sum_{J\in\mathcal{F}_1} \mu(IJ)^q}{\sum_{I\in\mathcal{F}_n}\sum_{J\in\mathcal{F}_1} \mu(IJ)^q} = \frac{\mu(I)^q(p_{n+1}^q + (1-p_{n+1})^q)}{\sum_{I\in\mathcal{F}_n}(p_{n+1}^q + (1-p_{n+1})^q)\mu(I)^q}$$

and therefore $\nu_{n+1}(I) = \nu_n(I)$ for all $I \in \mathcal{F}_n$. •

By remark 2.2, we then deduce that the Hausdorff and the packing dimension of ν are given by an entropy formula. In other terms, we have

$$\dim \nu = \liminf_{n\to\infty} -\tau'_{\nu,n}(1) = h_*(\nu)$$

and

$$\operatorname{Dim} \nu = \limsup_{n\to\infty} -\tau'_{\nu,n}(1) = h^*(\nu).$$

Now we cam prove Theorem 1.1.

Proof of Theorem 1.1 The upper bound is a well known fact of multifractal formalism (see for instance [BMP92]). In fact we have

1. If $\alpha \le -\tau'(0^+)$ then $\dim E_\alpha \le \dim \underline{E}_\alpha \le \tau^*(\alpha)$.
2. If $\alpha \ge -\tau'(0^-)$ then $\dim F_\alpha \le \dim \overline{F}_\alpha \le \tau^*(\alpha)$.
3. $-\tau'(0^+) \le \alpha \le -\tau'(0^-)$ then $\tau^*(\alpha) = \tau(0)$ and the upper bound is trivial.

Lemma 2.3 and a straightforward computation imply $\tau_{\nu,n}(s) = \tau_{\mu,n}(qs) - s\tau_{\mu,n}(q)$. Using once again the (inhomogeneous) Bernoulli property of μ and remark 2.2 we deduce that

$$-\tau'_\nu(1^+) = \liminf -\tau'_{\nu,n}(1) = \liminf\left(-q\tau'_{\mu,n}(q) + \tau_{\mu,n}(q)\right).$$

The following lemma then implies the lower bound.

Lemma 2.4 *We have* $\nu\left(\underline{E}_{-\tau'(q^-)} \cap \overline{F}_{-\tau'(q^+)}\right) = 1$.

Proof For $\eta > 0$ we put $\beta = -\tau'_\mu(q^-) + \eta$ and we prove that $\nu(\Sigma \setminus \underline{E}_\beta) = 0$; it can be shown in a similar way that $\nu(\Sigma \setminus \overline{F}_\gamma) = 0$ for $\gamma < -\tau'_\mu(q^+)$. The lemma then easily follows.

It suffices to show that $\Sigma \setminus E_\beta = \left\{x \in \Sigma\ ;\ \liminf_{k\to\infty} \alpha_n(x) > \beta\right\}$ is of 0 ν-measure. Consider the collection $\mathcal{R}_n(\beta)$ of cylinders $I \in \mathcal{F}_n$ satisfying $\dfrac{\log\mu(I)}{\log|I|} > \beta$. It is clear that $\Sigma \setminus E_\beta = \limsup_{n\to\infty} \mathcal{R}_n(\beta)$.

Let $(\tau_{\mu,n_k})_{k\in\mathbb{N}}$ be the subsequence of $(\tau_{\mu,n})_{n\in\mathbb{N}}$ such that $\lim_{k\to\infty}\tau_{\mu,n_k}(q)=\tau_\mu(q)$. Using the convergence of $\tau_{\mu,n_k}(q)$ we can choose (and fix) $t<0$ such that for k big enough

$$\tau_\mu(q+t)-\tau_{\mu,n_k}(q) < -\left(\beta-\frac{\eta}{2}\right)t = \left(\tau'_\mu(q^-)-\frac{\eta}{2}\right)t$$

We get $\mu(I)^{-t}|I|^{\beta t}\leq 1$ and hence

$$\begin{aligned}
\sum_{I\in\mathcal{R}_{n_k}(\beta)}\nu(I) &= \sum_{I\in\mathcal{R}_{n_k}(\beta)}\mu(I)^q|I|^{\tau_{\mu,n_k}(q)} = \sum_{I\in\mathcal{R}_{n_k}(\beta)}\mu(I)^{q+t}|I|^{\tau_{\mu,n_k}(q)-\beta t}\mu(I)^{-t}|I|^{\beta t}\\
&\leq \sum_{I\in\mathcal{R}_{n_k}(\beta)}\mu(I)^{q+t}|I|^{\tau_{\mu,n_k}(q)-\beta t} \leq \sum_{I\in\mathcal{F}_{n_k}}\mu(I)^{q+t}|I|^{\tau_\mu(q+t)-\frac{\eta}{2}t}\\
&\leq \sum_{I\in\mathcal{F}_{n_k}}\mu(I)^{q+t}|I|^{\tau_{\mu,n_k}(q+t)}=1,
\end{aligned}$$

where for the last inequality we used the fact that $\tau_\mu(q+t)=\limsup\tau_{\mu,n}(q+t)$. It easily follows that $\limsup_{k\to\infty}\sum_{I\in\mathcal{R}_{n_k}(\beta)}\nu(I)=0$ and the lemma is proved. •

The proof of Theorem 1.1 is now completed. •

Let f and g be the functions defined by $f(t)=\dim\underline{E}_t$ and $g(t)=\dim\overline{F}_t$. Obviously, f is increasing and g is decreasing. Recall that t is a non-stationary point of a monotone function h if $h(s)\neq h(t)$ for all $s\neq t$.

Since $E_\alpha=\underline{E}_\alpha\setminus\bigcup_{\beta<\alpha}\underline{E}_\beta$, we deduce from theorem 1.1 the following.

Remark 2.5 If $\alpha=-\tau'(q^-)$ for $q>0$ is a non-stationary point of f there a sequence of $q_m\leq q$ such that $\alpha_m=-\tau'(q_m^-)$ are non-stationary points of f converging to α and

$$\liminf_{n\to\infty}-q_m\tau'_{\mu,n}(q_m)+\tau_{\mu,n}(q_m)\leq\dim E_{\alpha_m}=\dim\underline{E}_{\alpha_m}\leq\tau^*(\alpha_m).$$

If $\alpha=-\tau'(q^+)$ for $q<0$ is a non-stationary point of g there a sequence of $q_m\geq q$ such that $\alpha_m=-\tau'(q_m^+)$ are non-stationary points of g converging to α and

$$\liminf_{n\to\infty}-q_m\tau'_{\mu,n}(q_m)+\tau_{\mu,n}(q_m)\leq\dim F_{\alpha_m}=\dim\overline{F}_{\alpha_m}\leq\tau^*(\alpha_m).$$

We conjecture that under the same conditions on α we should also have $\dim E_\alpha=\dim\underline{E}_\alpha$ ($\dim F_\alpha=\dim\overline{F}_\alpha$ respectively).

3 Some conditions ensuring the validity of multifractal formalism

In this section we prove Theorem 1.2. We will use the following result.

Proposition 3.1 *For $q > 0$ consider (τ_{μ,n_k}) the subsequence of $(\tau_{\mu,n})$ such that*

$$\lim_{k\to\infty} \tau_{\mu,n_k}(q) = \limsup_{n\to\infty} \tau_{\mu,n}(q).$$

Then if q is a differentiability point of τ_μ we have

$$\lim_{k\to\infty} \tau'_{\mu,n_k} = \tau'_\mu(q).$$

Proof The proposition is a immediate consequence of the following lemmas.

Lemma 3.2 *Under the assumptions of proposition 3.1*

$$\tau'_\mu(q^+) \geq \limsup_{k\to\infty} \tau'_{\mu,n_k}(q),$$

where $\tau'_\mu(q^+)$ stands for the right hand dérivative ot τ_μ at q.

On the other hand, we get

Lemma 3.3 *Under the assumptions of proposition 3.1*

$$\tau'_\mu(q^-) \leq \liminf_{k\to\infty} \tau'_{\mu,n_k}(q),$$

where $\tau'_\mu(q^-)$ stands for the left hand dérivative of τ_μ at q.

Proof of lemma 3.2. Take $\epsilon > 0$ and $\tilde{q} > q$ satisfying

$$\left| \frac{\tau_\mu(\tilde{q}) - \tau_\mu(q)}{\tilde{q} - q} - \tau'_\mu(q^+) \right| < \epsilon/8$$
$$|\tilde{q} - q| \sup_{n\in\mathbb{N}} ||\tau''_{\mu,n}||_\infty < \epsilon/8.$$

and consider $(\tilde{n}_k)$ such that $\lim_{k\to\infty} \tau_{\mu,\tilde{n}_k}(\tilde{q}) = \limsup_{n\to\infty} \tau_{\mu,n}(\tilde{q})$.

We can chose k big enough to have

$$\begin{aligned}
\frac{|\tau_{\mu,n_k}(q)-\tau_\mu(q)|}{|\tilde{q}-q|} &< \epsilon/8 \\
\frac{|\tau_{\mu,\tilde{n}_k}(\tilde{q})-\tau_\mu(\tilde{q})|}{|\tilde{q}-q|} &< \epsilon/8 \\
\tau_{\mu,n_k}(\tilde{q}) &\leq \tau_{\mu,\tilde{n}_k}(\tilde{q})+(\tilde{q}-q)\epsilon/8.
\end{aligned}$$

We then obtain

$$\begin{aligned}
\tau'_\mu(q^+) &\geq \frac{\tau_\mu(\tilde{q})-\tau_\mu(q)}{\tilde{q}-q}-\epsilon/8 \geq \frac{\tau_{\mu,\tilde{n}_k}(\tilde{q})-\tau_{\mu,n_k}(q)}{\tilde{q}-q}-\epsilon/4 \\
&\geq \frac{\tau_{\mu,n_k}(\tilde{q})-\tau_{\mu,n_k}(q)}{\tilde{q}-q}-3\epsilon/8 \geq \tau'_{\mu,n_k}(q)-|\tilde{q}-q|\sup_{n\in\mathbb{N}}||\tau''_{\mu,n}||_\infty-\epsilon/2 \\
&\geq \tau'_{\mu,n_k}(q)-\epsilon.
\end{aligned}$$

and the proof is completed. •

Lemma 3.3 is proven in a similar manner and together with lemma 3.2 provide the proposition's proof. •

We can now prove Theorem 1.2.

Proof of Theorem 1.2. Let ν be the Gibbs-measure defined in lemma 2.3. Since

$$\tau_{\nu,n}(s)=\tau_{\mu,n}(qs)-s\tau_{\mu,n}(q)$$

we get

$$\tau'_{\nu,n}(1)=q\tau'_{\mu,n}(q)-\tau_{\mu,n}(q).$$

Using the convergence of $\tau_{\mu,n}(q)$ we deduce from Proposition 3.1 that

$$\lim_{n\to\infty}\tau'_{\nu,n}(1)=\lim_{n\to\infty}\left(q\tau'_{\mu,n}(q)-\tau_{\mu,n}(q)\right)=q\tau'_\mu(q)-\tau_\mu(q).$$

Lemma 2.3 then implies that $\tau'_\nu(1)$ exists and

$$\dim\nu=\operatorname{Dim}\nu=-\tau'_\nu(1)=-q\tau'_\mu(q)+\tau_\mu(q).$$

On the other hand, for $I\in\mathcal{F}_n$, we have

$$\frac{\log\nu(I)}{\log|I|}=q\frac{\log\mu(I)}{\log|I|}+\tau_{\mu,n}(q)$$

Since

$$\lim_{n\to\infty} \frac{\log \nu(I_n(x))}{\log |I_n(x)|} = \dim \nu = \mathrm{Dim}\, \nu \ ; \nu\text{-a.s.}$$

we obtain that $\lim_{n\to\infty} \dfrac{\log \mu(I_n(x))}{\log |I_n(x)|} = -\tau_\mu'(q)$, ν-a.s. We conclude that

$$\dim (E_\alpha \cap F_\alpha) \geq \dim \nu = \tau_\mu^*(\alpha).$$

The opposite inequality being always valid, the proof is done. •

4 Phase transitions

Theorem 4.1 *Let τ be a convex combination of functions $\tau(p_i, .)$ where $0 < p_i \leq 1/2$, $i = 1, ..., n$. For any $1 < q_1 < q_2 < \infty$ there exists another convex combination $\tilde{\tau}$ of functions $\tau(p_j', .)$ such that*

- *$\tilde{\tau}(q_i) = \tau(q_i)$ and $\tilde{\tau}'(q_i) \neq \tau'(q_i)$, $i = 1, 2$,*
- *for $q \in (q_1, q_2)$, $\tilde{\tau}(q) > \tau(q)$,*
- *else, for $q \notin [q_1, q_2]$, $\tilde{\tau}(q) < \tau(q)$.*

5 Proof of Theorem 4.1

In this section whenever we use the notation p_i for a weight in $(0, 1)$ we will also note $\tau_i = \tau(p_i, .)$.

Lemma 5.1 *Take $\tau = \lambda\tau(p_1, .) + (1 - \lambda)\tau(p_2, .)$ with $0 < p_1 < p_2 < 1/2$ and $\lambda \in (0, 1)$. For $p_0 \in (0, 1/2)$ one of the following occurs :*

1. *either $\tau(q) \neq \tau(p_0, q)$, for all $q > 1$,*
2. *or, there exists $q_0 > 1$ such that $\tau(q) > \tau(p_0, q)$ for $q < q_0$ and $\tau(q) < \tau(p_0, q)$ for $q > q_0$. The point q_0 is, then, the unique point of equality between these functions.*

To prove this lemma we need the following subsidiary result.

Lemma 5.2 *Let $p_1 < p_2 < p_3$ take values in $(0, 1/2)$ and τ_1, τ_2, τ_3 be the functions $\tau(p_1, .), \tau(p_2, .), \tau(p_3, .)$ respectively. Then $\dfrac{\tau_1 - \tau_2}{\tau_2 - \tau_3}$ is decreasing on $(1, +\infty)$.*

Although the proof only uses elementary calculus, it is a litte bit "tricky" and canot be omitted.

Proof of Lemma 5.2 Taking into account the trivial equality

$$\tau(p', q) - \tau(p'', q) = \int_{p''}^{p'} \frac{\partial \tau}{\partial p}(p, q)dp$$

we only need to show that if $p' < p''$ then $\frac{\partial \tau}{\partial p}(p', q) : \frac{\partial \tau}{\partial p}(p'', q)$ is decreasing on $q \in (1, \infty)$. We get

$$\begin{aligned}\frac{\partial \tau}{\partial p}(p', q) : \frac{\partial \tau}{\partial p}(p'', q) &= \frac{1 - (-1 + 1/p')^{q-1}}{1 + (-1 + 1/p')^q} : \frac{1 - (-1 + 1/p'')^{q-1}}{1 + (-1 + 1/p'')^q} \\ &= p'' \frac{1 - {s_1}^{q-1}}{1 + {s_1}^q} : p' \frac{1 - {s_2}^{q-1}}{1 + {s_2}^q}\end{aligned}$$

where $s_1 = -1 + 1/p' > 1$ and $s_2 = -1 + 1/p'' > 1$.

If we set $f(s, q) = \ln \frac{1 - s^{q-1}}{1 + s^q}$, with $s, q > 1$, it is sufficient to prove that $\frac{\partial f}{\partial s} f(s, q)$ is decreasing in q. We calculate

$$\frac{\partial f}{\partial s} f(s, q) = \frac{(q - 1)s^{q-2}}{s^{q-1} - 1} - \frac{qs^{q-1}}{s^q + 1}.$$

We multiply by s and need to show that $\frac{(q - 1)s^{q-1}}{s^{q-1} - 1} - \frac{qs^q}{s^q + 1}$ is decreasing which is equivalent to $q - 1 + \frac{q-1}{s^{q-1}-1} - q + \frac{q}{s^q+1}$ being decreasing.

Put $Q = q - 1$; it remains to show that $\frac{q - 1}{s^{q-1} - 1} + \frac{q}{s^q + 1} = \frac{Q}{s^Q - 1} + \frac{Q}{s^{Q+1} + 1} + \frac{1}{s^{Q+1} + 1}$ decreases in $Q > 0$ and since the last term is decreasing it suffices to show that $\frac{Q}{s^Q - 1} + \frac{Q}{s^{Q+1} + 1}$ is doing the same. By taking derivatives we need to show that

$$(s^Q - 1)(s^{Q+1} + 1) - s^Q \ln s^Q (s^{Q+1} + 1) - s^{Q+1} \ln s^Q (s^Q - 1)$$

is negative for $Q > 0$, which is trivial since $s^Q \ln s^Q > s^Q - 1$. •

Proof of lemma 5.1. Let us first remark that τ and $\tau(p_0, .)$ can coincide at one point only if $p_0 \in (p_1, p_2)$. Moreover, $\tau(q) = \tau(p_0, q)$ implies

$$\frac{\tau(p_1, q) - \tau(p_0, q)}{\tau(p_0, q) - \tau(p_2, q)} = \frac{\lambda}{1 - \lambda}.$$

By lemma 5.2 this can only occur once. The lemma 5.1 easily follows on the decreasing property of the ratio. •

The following two lemmas prove Theorem 4.1 in the particular case $n = 2$.

Lemma 5.3 *Take $\lambda_1, \lambda_2 \in (0, 1)$ such that $\lambda_1 + \lambda_2 = 1$, $1 < p_1 < p_2 < 1/2$ and set $\tau = \lambda_1\tau_1 + \lambda_2\tau_2$. Fix $1 < q_1 < q_2 < +\infty$ and consider $p_1 < p_4 < p_2 < p_5 < 1/2$ such that $\tau(p_4, q) = \tau(q)$. Then there is a unique convex combination $\tilde{\tau}$ of τ_1, τ_4 and τ_5 such that*

$$\tilde{\tau}(q_1) = \tau(q_1) \text{ and } \tilde{\tau}(q_2) = \tau(q_2).$$

Furthermore, for $i = 1, 2$, we have $\tau'(q_i) \neq \tilde{\tau}'(q_i)$ and $\tau(q) \neq \tilde{\tau}(q)$ if $q \neq q_i$.

Proof It suffices to show that the linear system

$$\left\{ \begin{array}{lllll} \lambda_3\tau_1(q_1) & +\lambda_4\tau_4(q_1) & +\lambda_5\tau_5(q_1) & = & \tau(q_1) \\ \lambda_3\tau_1(q_2) & +\lambda_4\tau_4(q_2) & +\lambda_5\tau_5(q_2) & = & \tau(q_2) \\ \lambda_3 & +\lambda_4 & +\lambda_5 & = & 1 \end{array} \right. \qquad (\mathbf{S})$$

has a unique positive solution $(\lambda_3, \lambda_4, \lambda_5)$. The existence of a unique solution is easy to verify. Let us show that this solution is positive.

Since $\tau(q_1) = \tau_4(q_1)$, we have

$$\lambda_3\tau_1(q_1) + \lambda_5\tau_5(q_1) = (1 - \lambda_4)(\lambda_1\tau_1(q_1) + \lambda_2\tau_2(q_1))$$

which is equivalent to

$$\frac{\lambda_3}{\lambda_3 + \lambda_5}\tau_1(q_1) + \frac{\lambda_5}{\lambda_3 + \lambda_5}\tau_5(q_1) = \lambda_1\tau_1(q_1) + \lambda_2\tau_2(q_1). \qquad (4)$$

This implies that $\lambda_3\lambda_5 > 0$. Moreover, since $\tau_5 < \tau_2$, we also have $\frac{\lambda_3}{\lambda_3+\lambda_5} > \lambda_1$.

Let us show that λ_3 and λ_5 are positive. Otherwise, by the above remark, we have $\lambda_3 < 0$, $\lambda_5 < 0$ and $\lambda_4 > 0$. By the system (S) we have

$$\tau_4(q) = \frac{\lambda_1 - \lambda_3}{\lambda_4}\tau_1(q) + \frac{\lambda_2}{\lambda_4}\tau_2(q) - \frac{\lambda_5}{\lambda_4}\tau_5(q)$$

at the points $q = q_1$ and $q = q_2$. We then obtain that

$$\frac{\lambda_1 - \lambda_3}{\lambda_4}\frac{\tau_1 - \tau_4}{\tau_4 - \tau_2}(q) = \frac{\lambda_2}{\lambda_4} + -\frac{\lambda_5}{\lambda_4}\frac{\tau_4 - \tau_5}{\tau_4 - \tau_2}(q)$$

for $q = q_1$ and $q = q_2$. Since $p_1 < p_4 < p_2$, by Lemma 5.2 the function $\frac{\tau_1-\tau_4}{\tau_4-\tau_2}$ is decreasing. On the other hand, since $p_4 < p_2 < p_5$, Lemma 5.2 implies that the function $\frac{\tau_4-\tau_5}{\tau_4-\tau_2} = 1+\frac{\tau_2-\tau_5}{\tau_4-\tau_2}$ is increasing. Thus, these functions cannot coincide at two points so we conclude that λ_3 and λ_5 are positive.

Let us now prove that $\lambda_4 > 0$. By (4) we have

$$\frac{\lambda_3}{\lambda_3+\lambda_5}\tau_1(q_1) + \frac{\lambda_5}{\lambda_3+\lambda_5}\tau_5(q_1) = \lambda_1\tau_1(q_1) + \lambda_2\tau_2(q_1)$$

which gives that

$$\lambda_2\tau_2(q_1) = \left(\frac{\lambda_3}{\lambda_3+\lambda_5} - \lambda_1\right)\tau_1(q_1) + \frac{\lambda_5}{\lambda_3+\lambda_5}\tau_5(q_1).$$

Using Lemma 5.2, for $q > q_1$ we get

$$\lambda_2\tau_2(q) > \left(\frac{\lambda_3}{\lambda_3+\lambda_5} - \lambda_1\right)\tau_1(q) + \frac{\lambda_5}{\lambda_3+\lambda_5}\tau_5(q)$$

and

$$\lambda_1\tau_1(q) + \lambda_2\tau_2(q) > \frac{\lambda_3}{\lambda_3+\lambda_5}\tau_1(q) + \frac{\lambda_5}{\lambda_3+\lambda_5}\tau_5(q).$$

In particular, for $q = q_2$ we find that

$$\lambda_3\tau_1(q_2) + \lambda_5\tau_5(q_2) + \lambda_4\tau(q_2) < \tau_(q_2) = \lambda_3\tau_1(q_2) + \lambda_4\tau_4(q_2) + \lambda_5\tau_5(q_2)$$

and we deduce that

$$\lambda_4\tau(q_2) < \lambda_4\tau_4(q_2).$$

It follows from Lemma 5.2 $\lambda_4 > 0$.

The last assertion follows directly from the independancy of the vector families

$$\left\{\begin{pmatrix}\tau_1(q_1)\\ \tau_4(q_1)\\ \tau_5(q_1)\end{pmatrix}, \begin{pmatrix}\tau_1(q_2)\\ \tau_4(q_2)\\ \tau_5(q_2)\end{pmatrix}, \begin{pmatrix}\tau_1'(q_i)\\ \tau_4'(q_i)\\ \tau_5'(q_i)\end{pmatrix}\right\}$$

and

$$\left\{\begin{pmatrix}\tau_1(q_1)\\ \tau_4(q_1)\\ \tau_5(q_1)\end{pmatrix}, \begin{pmatrix}\tau_1(q_2)\\ \tau_4(q_2)\\ \tau_5(q_2)\end{pmatrix}, \begin{pmatrix}\tau_1(q)\\ \tau_4(q)\\ \tau_5(q)\end{pmatrix}\right\},$$

which can be easily established. •

Lemma 5.4 *The functions τ and $\tilde{\tau}$ defined in lemma 5.3 verify $\tilde{\tau}(q) > \tau(q)$ if and only if $q \in (q_1, q_2)$.*

Proof Let us first remark that for λ_3, λ_4 and λ_5 defined by the linear system (**S**) we have $\dfrac{\lambda_3}{1-\lambda_4}\tau_1(q_1) + \dfrac{\lambda_5}{1-\lambda_4}\tau_5(q_1) = \tau(q_1) = \tau_4(q_1)$. Put $\rho = \frac{\lambda_3}{\lambda_3+\lambda_5}\tau_1 + \frac{\lambda_5}{\lambda_3+\lambda_5}\tau_5$ and consider the function $\Lambda : [0,1] \to \mathcal{C}^\infty([1,\infty),\mathbb{R})$ that assigns $\mu \in [0,1]$ to $\Lambda(\mu) = \mu\tau_4 + (1-\mu)\rho$. Let us also take $q_1 = q_2$ so that

$$\Lambda(\lambda_4)'(q_1) = \tau'(q_1) \tag{5}$$

(the parameter λ_4 depends on q_2). It is sufficient to show that $\Lambda(\lambda_4) \leq \tau$: to obtain that $\Lambda(\lambda_4) \leq \tau$ outside $[q_1, q_2]$, for $q_1 < q_2$, one can use a simple continuity argument on the graph of $\Lambda(\lambda_4)$, seen as a function of q_2.

By (5) we obtain $\dfrac{(\tau-\rho)'(q_1)}{(\tau_4-\rho)'(q_1)} = 1 > \lambda_4$. The function $\dfrac{\tau-\rho}{\tau_4-\rho}$ being increasing in a neighborhood of q_1 (as we will show below) we get that for $q > q_1$ $(\tau-\rho)(q) > \lambda_4(\tau_4-\rho)(q)$ which implies $\Lambda(\lambda_4)(q) < \tau(q)$ for $q \neq q_1$.

To finish the proof we need to show that $\dfrac{\tau-\rho}{\tau_4-\rho}$ is increasing. Put $s_1 = \frac{\lambda_3}{\lambda_3+\lambda_5} - \lambda_1$, $k_5 = \frac{\lambda_5}{\lambda_3+\lambda_5} = 1 - k_1$ and $k_1 = \frac{\lambda_3}{\lambda_3+\lambda_5}$, all positive. We can write :

$$\begin{aligned} \frac{\rho-\tau}{\rho-\tau_4} &= \frac{s_1(\tau_1-\tau_2)+k_5(\tau_5-\tau_2)}{k_1(\tau_1-\tau_4)+k_5(\tau_5-\tau_4)} = \\ &= \frac{1}{1-\lambda_1}\frac{s_1 - \dfrac{\tau_2-\tau_5}{\tau_1-\tau_5}}{k_1 - \dfrac{\tau_4-\tau_5}{\tau_1-\tau_5}} = \frac{k_1}{s_1(1-\lambda_1)}\frac{1-f}{1-g} \end{aligned}$$

where f, g are both positive increasing functions by lemma 5.2. Moreover f/g is increasing which implies $f'g - g'f > 0$ and $g'(q_1) = f'(q_1)$ hence $(f-g)'(g+f) + g' - f' < 0$. The result follows. •

The proof of theorem 4.1 in the case $n > 2$ is now easy to derive : suppose $\tau = \sum_{k=1}^n \lambda_k \tau(p_k, .)$ and let $\tau(p_1, .)$ and $\tau(p_2, .)$ be the first two functions of the convex combination. Be the previous two lemmas there exist a convex combination $\hat{\tau}$ of three $\tau(p_i, .)$ functions such that

1. $\frac{1}{\lambda_1+\lambda_2}(\lambda_1\tau_1(q_i) + \lambda_2\tau_2(q_i)) = \hat{\tau}(q_i)$, for $i = 1, 2$
2. $\frac{1}{\lambda_1+\lambda_2}(\lambda_1\tau_1'(q_1) + \lambda_2'\tau_2(q_1)) < \hat{\tau}'(q_1)$, $\frac{1}{\lambda_1+\lambda_2}(\lambda_1\tau_1'(q_2) + \lambda_2'\tau_2(q_2)) > \hat{\tau}'(q_2)$
3. $\frac{1}{\lambda_1+\lambda_2}(\lambda_1\tau_1 + \lambda_2\tau_2) < \hat{\tau}$ on (q_1, q_2) and $\frac{1}{\lambda_1+\lambda_2}(\lambda_1\tau_1 + \lambda_2\tau_2) > \hat{\tau}$ on $(1,\infty) \setminus [q_1, q_2]$.

The function $\tilde{\tau} = (\lambda_1+\lambda_2)\hat{\tau} + \sum_{k=3}^n \lambda_k\tau(p_k, .)$ satisfies then the conclusion of theorem 4.1.

We can now prove theorem 1.3 :

There exists an inhomogeneous Bernoulli product μ such that the spectrum τ of μ is not derivable on a dense subset of $[1,\infty)$.

The strategy of the demonstration of this theorem is the following : we first find inhomogeneous Bernoulli products that are not derivable at a finite number of predefined points and we construct the measure μ using Cantor's diagonal argument.

Lemma 5.5 *For any $p_1,...,p_n$ and any convex combination τ of $\tau(p_1,.),...,\tau(p_n,.)$ there exist an inhomogeneous Bernoulli measure μ whose multifractal spectrum equals τ.*

The proof of this lemma is not difficult and left to the reader. Let un now prove theorem 1.3.

Proof of Theorem 1.3. Fix $(q_n)_n$ a sequence of real numbers, dense in $[1,\infty)$ and nested in the sense that $q_{2n+1} < q_{2n+2}$ and $\{q_1,...,q_{2n}\} \cap [q_{2n+1}, q_{2n+2}] = \emptyset$ for all $n \geq 0$. Let $p_1, p_2 \in (0,1)$ and $\tau_1 = \frac{1}{2}\tau(p_1,.) + \frac{1}{2}\tau(p_2,.)$. By the previous lemma we can construct a Bernoulli product μ_1 of spectrum τ_1. Theorem 4.1 implikes then the existence of a convex combination τ_2 of $\tau(p_i,.)$'s functions, such that

1. $\tau_1(q_i) = \tau_2(q_i)$, for $i = 1,2$, $\tau_1'(q_1) < \tau_2'(q_1)$, $\tau_1'(q_2) > \tau_2'(q_2)$
2. $\tau_2 > \tau_1$ on (q_1,q_2) and $\tau_2 < \tau_1$ on $(1,\infty) \setminus [q_1,q_2]$.

We can therefore define a measure μ_2 of spectrum τ_2 ; Using μ_1 and μ_2 we can construct a measure ν_2 of spectrum $\max\{\tau_1,\tau_2\}$: Let μ_i me the inhomogeneous Bernoulli measure of spectrum τ_i, $i = 1,2$. Take $(\ell_k)_k$ a sequence of integers such that $\dfrac{\ell_{k+1}}{\sum_1^k \ell_i} \to \infty$. On dyadique intervalles of length between $2^{-\ell_{2k}}$ and $2^{-\ell_{2k+1}}$ apply the weight distribution of μ_1 and on dyadique intervalles of length between $2^{-\ell_{2k+1}}$ and $2^{-\ell_{2k+2}}$ apply the weight distribution of μ_2, where $k \in \mathbb{N}$. It is easy to verify that the resulting inhomogeneous measure ν_2 has spectrum $\max\{\tau_1,\tau_2\}$. Remark that this spectrum equals τ_2 on $[q_1,q_2]$ and τ_1 elsewhere on $[1,\infty)$.

We proceed by induction. Suppose the measures $\nu_1 = \mu_1$, μ_2, ν_2,..., μ_n, ν_n defined and denote τ_i the spectrum of the measure μ_i, $i \in \{1...,n\}$. We assume that that on every interval $[q_{2i+1}, q_{2i+2}]$, where $i \leq n$, the spectrum of ν_n equals $\max\{\tau_1,...\tau_n\}$ and is realized by τ_i. Let us construct μ_{n+1} and ν_{n+1}. Consider τ_j the function that equals the $\max\{\tau_1,...\tau_n\}$ on $[q_{2(n+1)+1}, q_{2(n+1)+2}]$. By theorem 4.1 we can find a function τ_{n+1} satisfying :

1. $\tau_{n+1}(q_{2(n+1)+i}) = \tau_j(q_{2(n+1)+i})$, for $i = 1,2$,
   $\tau_{n+1}'(q_{2(n+1)+1}) > \tau_j'(q_{2(n+1)+1})$, $\tau_{n+1}'(q_{2(n+1)+2}) < \tau_j'(q_{2(n+1)+2})$
2. $\tau_{n+1} > \tau_j$ on $(q_{2(n+1)+1}, q_{2(n+1)+2})$ and $\tau_{n+1} < \tau_j$ on $(1,\infty) \setminus [q_{2(n+1)+1}, q_{2(n+1)+2}]$.

Let μ_{n+1} be the inhomogeneous Bernoulli measure of spectrum τ_{n+1}. To define the measure ν_{n+1} we use the previous procedure convenably adapted : Take $(\ell_k)_k$ a sequence of

integers such that $\dfrac{\ell_{k+1}}{\sum_1^k \ell_i} \to \infty$. On dyadique intervalles of length between $2^{-\ell_{(n+1)k+i}}$ and $2^{-\ell_{(n+1)k+i+1}}$ apply the weight distribution of μ_i, where $i = 1, ..., n+1$, $k \in \mathbb{N}$. It is easy to verify that the resulting inhomogeneous measure ν_{n+1} has spectrum $\max\{\tau_1, ...\tau_{n+1}\}$ on $(1, \infty)$. Remark that this spectrum equals τ_{n+1} on $[q_{2(n+1)+1}, q_{2(n+1)+2}]$ and $\max\{\tau_1, ...\tau_n\}$ elsewhere on $[1, \infty)$.

To end the proof we use Cantor's diagonal argument : take $(\ell_k)_k$ comme ci-dessus et define the measure ν by applying the weight distribution of ν_k on dyadique intervalles of length between $2^{-\ell_k}$ and $2^{-\ell_{k+1}}$. The spectrum of the measure ν equals then $\tau = \sup_{n\in\mathbb{N}} \tau_n$. By construction the function τ is not derivable at the points $(q_k)_k$ and the proof of theorem 1.3 is complete. •

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